Report on the Virasoro Operators in String Theory

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Abstract

In this paper we discuss the role of the Virasoro operators in bosonic string theory. We give a streamlined explanation of the origin of the Virasoro operators in classical theory, before quantising the theory using the old covariant quantisation approach to motivate the full Virasoro algebra. Following this we briefly discuss the interpretations of the algebra on the classical and quantum levels. We derive the physical state conditions and explain the use of the Virasoro operators in their construction, before explicitly using them to compute the low lying levels of the open string spectrum. Finally we explain the role of the Virasoro operators in the theory of vertex operators. We construct, in detail, the first two vertex operators corresponding to the tachyon and massless vector meson state for the open string, complete with the conditions they must obey, in order to illustrate the correspondence between physical states and vertex operators.

1 Introduction

This report aims to give an account of the role of the Virasoro operators in string theory. The Virasoro generators are a set operators that form an infinite dimensional Lie algebra, and they appear a number of times in mathematics and theoretical physics. The significance of these operators can be observed by the foundational role they play in the development of bosonic string theory. Understanding the Virasoro group is of central importance in string theory, since the gauge invariance of the theory is a consequence of the conformal invariance of the two-dimensional world sheet field theory which the Virasoro operators generate.

The Virasoro operators permeate a great number of areas in string theory. Here we discuss the use of the Virasoro operators in the most basic string theory; the bosonic string propagating in Minkowski space-time. For the majority of this report we focus on the open string. We will give a streamlined account of the ideas that led to the Virasoro generators followed by a more detailed discussion of the role they play in particular areas. First we will give the basics of the classical string theory and describe the mode expansion. Following this we explain how the Virasoro generators turn up in classical string theory before quantising, using the old covariant quantisation procedure, and giving the full Virasoro algebra. After this we discuss interpretations of the classical and quantum algebra. We then explain what the physical state conditions are. We will show how the physical state conditions connect to the classical constraints and how the Virasoro operators encode these. Using the physical state conditions we then determine the spectrum of the open bosonic string up to level two. Finally we consider vertex operators. We explain the how the Virasoro operators determine the form of these operators and, using the old covariant quantisation approach, compute the vertex operators for the open string corresponding to the level zero and level one state. In reference [4] a didactic exposition of string theory can be found. We will adopt this as our main reference text on the subject.

The Virasoro operators play a fundamental role in string theory and as such it is difficult to overstate its importance. Some might say it is the most important algebra in the theory. In the same light, it is difficult to give a brief account of the many areas in which it plays a part, since the mathematical structure can be found underpinning the development of almost everything. However, in the following sections we will try to give a detailed and concise exposition of the subject.

2 The Mode Expansion

In this section we outline the mode expansion of the coordinates of the open string, a result which is repeatedly used in later sections. This and the next few sections establish basic results and we use [4] and [7] as our main references. We begin by describing the Virasoro operators as they appear in the analysis of the classical string. The motion of the classical string can be described by the Brink-Di Vecchia-Howe free string action, otherwise known as the Polyakov action,

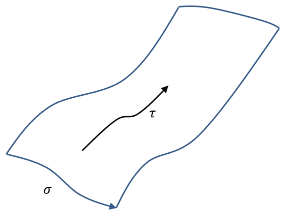

Figure 1: An illustration of the world-sheet traced out by the open string

$$S = -\frac{T}{2}\int_M d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{1}$$

There are various terms in this action that may need some explaining. $\sigma^\alpha = (\tau, \sigma)$ denote the parameters which intrinsically label points of the world-sheet manifold that the string sweeps out in D-dimensional Minkowski space-time. An illustration of this is provided in figure 1. The integration is carried out over this surface. The functions X^μ describe the embedding of the string in this target space. The term $h^{\alpha\beta}$ and $\sqrt{-h}$ denotes an auxiliary field that determines the intrinsic geometry of the two-dimensional world-sheet and the magnitude of its determinant. Following Polyakov, both the free field coordinate functions and the auxiliary metric are to be varied independently.

The Polyakov action contains a number of symmetries:

- Poincaré invariance: Under Lorentz transformations and space-time translations of the string coordinates the action remains unchanged. This is a global symmetry from the perspective of the world-sheet.
- Reparameterisation invariance: If we make the transformation $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ the action remains unchanged.
- Weyl invaraince: If we make the transformations $X^\mu \to X^\mu(\sigma)$ and $h_{\alpha\beta} \to \Omega^2(\sigma)h_{\alpha\beta}(\sigma)$ the action remains unchanged.

It is important to recognise that the last two symmetries represent gauge redundancies. Using these symmetries we can make a choice of gauge for the field $h_{\alpha\beta}$, and a particularly nice one is the conformal gauge

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \tag{2}$$

We are free to choose this gauge due to the gauge invariance previously described. This choice manifestly preserves the Poincaré invariance of the theory. However in light-cone coordinates ($\sigma^\pm = \tau \pm \sigma$) the invariant line element becomes

$$ds^2 = -d\sigma^+ d\sigma^- \tag{3}$$

Thus, the gauge choice fixes the gauge redundancy up to analytic transformations

$$\sigma^{\pm} \to f_{\pm}(\sigma^{\pm}) \tag{4}$$

and global Weyl transformations

$$h_{\alpha\beta}(\sigma) \to \text{const} \cdot h_{\alpha\beta}(\sigma) \tag{5}$$

This is a residual gauge redundancy which remains once we have specified the conformal gauge and is connected with the Virasoro operators.

Once we have chosen the gauge the action becomes

$$S = -\frac{T}{2} \int_M d^2\sigma \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \tag{6}$$

By varying the action with respect the fields X^μ and setting this equal to zero we obtain the equations of motion satisfied by the fields

$$\frac{\delta S}{\delta X^\mu} = \left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X_\mu = 0 \tag{7}$$

By taking Neumann boundary conditions ($X'(\tau, \sigma = 0, \pi) = 0$) this leads to the classical mode expansion for the open string

$$X^\mu(\sigma,\tau) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m /= 0} \frac{1}{m} \alpha_m^\mu e^{-im\tau} \cos m\sigma \tag{8}$$

where x^μ, p^μ correspond to the centre of mass and momentum of the string and l_s, α_m^μ correspond to the string length and a general Fourier mode of the string. In many texts one finds that the string length is defined in terms of the Regge slope, α', such that $l_s^2 = 2\alpha'$. Usually this constant is then set to one. In the following report we will keep all factors of l_s.

3 The Classical Constraints

In this section we establish the classical constraints imposed on the motion of the string. By varying the action (6) with respect to the auxiliary field $h_{\alpha\beta}$ one obtains an equation of motion for a quantity which we identify as the energy-momentum tensor.

$$T_{\alpha\beta} = -\frac{2}{T} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu = 0 \tag{9}$$

This is a symmetric tensor which has a trace which is identically zero. Thus there are only two independent components, which give the constraints on motion of the string.

$$\dot{X}^2 + X'^2 = 0, \quad \dot{X} \cdot X' = 0 \tag{10}$$

If we transform to light-cone coordinates ($\sigma^\pm = \tau \pm \sigma$) we can write the constraints as

$$
\begin{aligned}
T_{++} &= \frac{1}{2}(T_{00} + T_{01}) = \partial_+ X \cdot \partial_+ X = 0 \\
T_{--} &= \frac{1}{2}(T_{00} - T_{01}) = \partial_- X \cdot \partial_- X = 0
\end{aligned}
\tag{11}
$$

The Virasoro generators are given by the Fourier modes of the energy-momentum tensor as follows

$$
L_m = \frac{T}{2} \int_0^{\pi} (e^{im\sigma} T_{++} + e^{-im\sigma} T_{--}) d\sigma = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{m-n} \cdot \alpha_n \tag{12}
$$

Thus at the classical level, through the vanishing of the energy-momentum tensor, the Virasoro generators classically encode an infinite number of constraint equations which the string must satisfy

$$
L_m = 0, \quad \forall m \in \mathbb{Z} \tag{13}
$$

One can show that the Lie bracket of two such operators is given by the following expression

$$
[L_m, L_n] = (m - n) L_{m+n} \tag{14}
$$

This is the Witt algebra.

4 Passing to the Quantum Theory

In this section we outline how one passes from the classical theory of the open string to the quantum theory. This section is based in part on [4] and [9]. In the old covariant approach to quantisation of the string, which we use for the majority of this report, the fields X^μ are treated as generalised coordinate operators satisfying the canonical commutation relations. The constraint equations are then imposed to determine the physical states.

Another way of proceeding with quantisation is through lightcone gauge quantisation. Here one identifies that, through the residual gauge symmetry (4), one can transform the world-sheet coordinates such that, in light-cone coordiantes, they satisfy the same wave-equation that the string coordinates do in (7). One then fixes the gauge symmetry by making the light-cone world-sheet coordinate equal to a particular light-cone string coordinate. Here the Lorentz covariance is explicitly broken and one has to check that the generators of the resulting Lorentz algebra satisfy the appropriate commutation relations, which requires the space-time dimension $D = 26$. The main advantage of this approach is that there are no constraints so we can immediately construct the spectrum of the string. We will not proceed with this however (see [4]).

Together with the conjugate momentum $P^\mu = T\dot{X}^\mu$ we can write the canonical commutation relations satisfied by the fields X^μ:

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = i\delta(\sigma-\sigma')\eta^{\mu\nu}$$
$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = [P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = 0 \tag{15}$$

Using our mode expansion (8), these translate into the following commutation relations

$$[x^\mu, p^\nu] = i\eta^\mu\nu, \quad [\alpha_n^\mu, \alpha_m^\nu] = n\eta^\mu\nu\delta_{m+n,0} \tag{16}$$

together with the hermiticity conditions

$$(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu, \quad (x^\mu)^\dagger = x^\mu \tag{17}$$

We defined the Virasoro operators at the classical level in (12). We can then interpret the α_m^μ as oscillator creation and annihilation operators for negative and positive m respectively. In common with the quantum formulation of many classical systems, once we pass to the quantum theory we encounter an operator ordering problem. We remedy this by using normal ordering, which places all lowering operators to the right and all raising operators to the left. The Virasoro operators are then defined as:

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} :\alpha_{m-n}\cdot\alpha_n: \tag{18}$$

Clearly the ordering only makes a difference if we have terms containing α_n^μ and α_{-n}^μ (and if $n \leq 0$). In fact, the only L_n which requires normal ordering is L_0. There is no clear way to resolve the ordering ambiguity, but we know that commuting these operators could, in principle, produce a constant. We go about remedying this situation by defining L_0 as the normal ordered expressions

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-m}\cdot\alpha_m \tag{19}$$

The ordering ambiguity is then relegated to the constraint equations for L_0, where we include an arbitrary normal ordering constant, a, i.e.

$$L_0 \to L_0 - a \tag{20}$$

Using these expressions it is straightforward to derive an expression for the commutator of the Virasoro operators with the oscillators:

$$[L_m, \alpha_n^\nu] = -n\alpha_{m+n}^\nu \tag{21}$$

By powering through a mass of commutators one can then show that we obtain the following for the quantised Virasoro operators:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n,0} \tag{22}$$

where, as we would have expected, we have an anomalous term corresponding to ambiguity that is present in L_0. This is the Virasoro algebra, the central extension of the Witt algebra (14).

5 Interpreting the Operators

In this section we review what the interpretation of the Virasoro algebra, with and without the central extension, correspodns to. This section is based in part on [5]. Without the central extension the Virasoro algebra, otherwise known as the Witt algebra, can be defined as the Lie algebra of complex vector fields on the circle S^1. An arbitrary element, X, in this Lie algebra is given by

$$X = f(z)\frac{\partial}{\partial z} = \sum_{n\in\mathbb{Z}} a_n e^{in\phi} i \frac{\partial}{\partial \phi} \tag{23}$$

A basis for these vector fields is thus given by

$$L_n = ie^{in\phi}\frac{\partial}{\partial \phi} \tag{24}$$

These operators can be show to satisfy the Witt algebra, given in equation (14). The physical reason for the connection between this and the classical Virasoro algebra as we defined it is due to the fact that we can represent the generators of transformations that preserve the conformal gauge in an analogous form to (24), using light-cone coordinates. Upon quantisation the light-cone coordinates effectively become periodic due to their appearance in the equations of motion only occurring in factors of $\exp in\sigma^{\pm}$, cementing the correspondence. The full algebra is given by the central extension of $\mathrm{Diff}(S^1)$, where the c-number term arises through the need to regularise products of operators.

The Virasoro operators generate world-sheet reparametrisations that preserve the conformal gauge condition (2) of the covariant formalism of the string theory we have developed. To demonstrate this, consider the commutator of the Virasoro operators with the string coordinates, whose mode expansion is given in (8).

$$\begin{aligned}
[L_m, X^\mu(\tau,\sigma)] &= [L_m, x_0^\mu] + l_s[L_m, \alpha_0^\mu]\tau + il_s \sum_{n/=0} \frac{1}{n} e^{-in\tau} \cos n\sigma [L_m, \alpha_n^\mu] \\
&= -il_s\alpha_m^\mu - il_s \sum_{n/=0} \cos n\sigma e^{-in\tau} \alpha_{m+n}^\mu \\
&= -il_s \sum_{n\in\mathbb{Z}} \cos n\sigma e^{-in\tau} \alpha_{m+n}^\mu \\
&= -\frac{il_s}{2} \sum_{n\in\mathbb{Z}} \left(e^{-in(\tau-\sigma)} + e^{-in(\tau+\sigma)} \right) \alpha_{m+n}^\mu
\end{aligned} \tag{25}$$

By changing variables in the first summation, $n \to n - m$, we then have

$$\begin{aligned}
[L_m, X^\mu(\sigma,\tau)] &= -\frac{il_s}{2}\sum_{n\in\mathbb{Z}}\left(e^{-i(n-m)(\tau-\sigma)} + e^{-i(n-m)(\tau+\sigma)}\right)\alpha_n^\mu \\
&= -\frac{il_s}{2}e^{im(\tau-\sigma)}\sum_{n\in\mathbb{Z}} e^{-in(\tau-\sigma)}\alpha_n^\mu - \frac{il_s}{2}e^{im(\tau+\sigma)}\sum_{n\in\mathbb{Z}} e^{-in(\tau+\sigma)}\alpha_n^\mu \\
&= -\frac{i}{2}e^{im(\tau-\sigma)}\left(\dot{X}^\mu - X'^\mu\right) - \frac{i}{2}e^{im(\tau+\sigma)}\left(\dot{X}^\mu + X'^\mu\right) \\
&= -ie^{im\tau}\cos m\sigma \dot{X}^\mu + e^{im\tau}\sin m\sigma X'^\mu \\
&= \xi_m^\alpha \partial_\alpha X^\mu
\end{aligned} \tag{26}$$

We can identify this with an infinitesimal coordinate transformation $\sigma^\alpha \to \sigma^\alpha + \xi_m^\alpha(\sigma,\tau)$, which generates the following transformation

$$\begin{aligned}
X^\mu(\tau + \epsilon\xi_m^\tau, \sigma + \epsilon\xi_m^\sigma) &= X^\mu(\tau,\sigma) + \xi_m^\alpha\partial_\alpha X^\mu + \mathcal{O}(\xi^2) \\
&= X^\mu(\tau,\sigma) + [L_m, X^\mu(\tau,\sigma)] + \mathcal{O}(\xi^2)
\end{aligned} \tag{27}$$

Thus the action of the Virasoro operators on the string coordinates generates the same change that would occur as a result of reparametrisation of the world-sheet. However, these aren't just ordinary reparametrisations. As we previously stated, they are reparametrisations that preserve the conformal gauge condition, and so they satisfy the following relation

$$\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha = \Lambda\eta^{\alpha\beta} \tag{28}$$

In essence, the residual gauge symmetries (28) that are present after choosing the conformal gauge are generated by the Virasoro operators.

6 The Physical State Conditions

In this section we establish the physical state conditions for the open bosonic string. The following is based, in part, on [4]. From the relations (16) and (17) it is clear that we can interpret the α_m^μ operators as lowering and raising operators for positive and negative m respectively. We define the string vacuum state, $|0\rangle$, as the state which satisfies

$$\alpha_m^\mu |0\rangle = 0, \quad m \geqslant 0 \tag{29}$$

and we can build a Fock space of states, spanned by the basis

$$|\{\lambda\};k\rangle = \prod_{n>0}\prod_{\mu_n=0}^{D-1}(\alpha_{-n}^\mu)^{\lambda_{n,\mu_n}}|0;k\rangle \tag{30}$$

where $|0;k\rangle$ denotes the string vacuum state with momentum k^μ. However, the commutation relations given in equation (16) hint at a sickness in the theory. There are a collection of states, such as $\alpha_\mu^0 |0\rangle$, that possess negative norm. These states are called ghosts. In order for the theory to be physically sensible, it is essential that all physical states have positive norm since ghosts in the physical spectrum of an interacting theory would lead to violations of causality and unitarity.

This is to be contrasted with what occurs in the light-cone gauge quantisation approach where one can construct the physical states by acting on the vacuum state with the set of oscillators $\{\alpha_n^i | i \in [1, 25]\}$. In this approach one can construct a manifestly ghost free spectrum (albeit not manifestly Lorentz covariant one). Naturally we would expect the old covariant approach to give identical results.

Thankfully this flaw isn't fatal, since we still have the constraints to impose, given (classically) by (13). We then take the physical space of allowed string states to be a subspace of the complete Fock space, one that is free from negative norm states and satisfies the constraints. In other words we don't impose all of these constraints as operator equations on the Hilbert space of states, but require that they have vanishing matrix elements in the portion of the Hilbert space corresponding to physical states

$$\langle \phi' | L_n - a\delta_{n,0} |\phi\rangle = 0 \tag{31}$$

Unfortunately we can't implement the constraints as $L_n = 0$ for all n on quantum mechanical states since this would lead to an inconsistency, for example:

$$\langle \phi | [L_n, L_{-n}] |\phi\rangle = \langle \phi | 2nL_0 |\phi\rangle + \frac{D}{12}(n^3 - n) \langle \phi|\phi\rangle \tag{32}$$

The most we can ask is that on physical states equation (30) is satisfied for $n \geqslant 0$. Since $L_{-n}^\dagger = L_n$ this can be replaced by the following set of conditions

$$(L_n - a\delta_{n,0}) |\phi\rangle = 0, \quad n \geqslant 0 \tag{33}$$

These conditions are called the Virasoro constraints or the physical state conditions. The constraint for $n = 0$ corresponds to a generalised mass-shell condition for physical states and the constraints for $n \geqslant 1$ corresponds to an infinity of gauge conditions. Furthermore, since the subalgebra $\{L_n, n \geqslant 3\}$ is generated by the operators L_1 and L_2, (e.g. L_3 $[L_2, L_{-1}]$) the constraints for $n \geqslant 1$ can be replaced by the following two conditions

$$L_1 |\phi\rangle = 0, \quad L_2 |\phi\rangle = 0 \tag{34}$$

We see that the Virasoro operators and algebra play a crucial role in determining physical states for the bosonic string. However, while the Virasoro conditions cut out a subspace which corresponds to physical states, we are not completely safe. In fact, for a consistent ghost free spectrum, the values of the normal ordering anomaly, a, and the space-time dimension, D, must be constrained to satisfy the following conditions:

$$a = 1, \quad D = 26 \qquad \text{or} \quad a \leqslant 1, \quad D \leqslant 25 \tag{35}$$

This is a statement of the no-ghost theorem, which was first proved by Brower and Goddard and Thorn, [1]. Throughout this report we will operate under the assumption that $a = 1$ and $D = 26$.

7 The Spectrum

In this section we construct the low lying positive norm physical spectrum. This work is based on the discussions given in [9], [7] and [2]. In section 6 we gave the Virasoro constraints that determine the physical state conditions for an excitation of the bosonic string. To find the physical state spectrum of the open string in Minkowski space-time we have to solve for all possible linear combinations of creation operators acting on a vacuum state, subject to these physical state conditions. Furthermore, since the Virasoro generators commute with the generators of the Poincaré group, we will find that the physical states form Lorentz multiplets.

Firstly, the level operator, R, is defined by

$$R = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = L_0 - \frac{1}{2}p^2 \tag{36}$$

This, and the momentum operator $\alpha_0^\mu = l_s p^\mu$, obey the following conditions

$$[L_m, p^\mu] = 0, \quad [L_0, R] = 0 \tag{37}$$

which means that the eigenstates of momentum are consistent with the Virasoro constraitns and we can label states by their eigenvalues with respect to both the level operator and the momentum operator. Furthermore, due to the hermiticity of p^μ and R, states corresponding to different eigenvalues are orthogonal to each other.

The action of the level operator on the general state, given in equation (30), is

$$R\,|\{\lambda\}; k\rangle = \big(\sum_n n\lambda_n\big)\,|\{\lambda\}; k\rangle \tag{38}$$

Thus the eigenvalue of R is a non-negative integer, $N = \sum_n n\lambda_n$. We call this eigenvalue the level of the state, and use it to categorise the excitations of the string. The L_0 physical state condition given in (32) can be translated into a generalised mass shell condition, so at level N a physical state has mass

$$m^2 = -k_\mu k^\mu = 2l_s^{-2}(N-1) \tag{39}$$

where k^μ is the momentum of the state under consideration. Now we construct the covariant physical states for levels $N = 0, 1, 2$ and, in part, 3:

(i) $N = 0$: At level zero we have the tachyon state, which we define as

$$|T(k)\rangle = |0; k\rangle \tag{40}$$

Equation (38) gives $m^2 = -k^2 = -2l_s^{-2}$. The rest of the Virasoro constraints are trivially satisfied. This is a singlet state of the Poincaré group. One interesting point about the tachyon state is that, at first sight, the presence of the tachyon itself may seem to be a problem. However this might suggest that the bosonic string, as we have formulated, is not in the correct vacuum state. For example a celebrated tachyonic field occurs in the Standard Model of particle physics, the Higgs field, and its shift to the true vacuum state of the theory gives mass to most of the particles in the Standard Model, including the part of the Higgs field which acquires a positive mass.

(ii) $N = 1$: At level one we have the massless state, with polarisation vector ζ:

$$|P(\zeta, k)\rangle = \zeta \cdot \alpha_{-1} |0; k\rangle \tag{41}$$

The mass shell condition gives $m^2 = -k^2 = 0$. The L_1 condition gives

$$\begin{aligned} L_1 |P(\zeta, k)\rangle &= L_1 \eta_{\mu\nu} \zeta^\mu \alpha^\nu_{-1} |0; k\rangle \\ &= \eta_{\mu\nu} \zeta^\mu [L_1, \alpha^\nu_{-1}] |0; k\rangle \\ &= l_s \zeta \cdot k |0; k\rangle \\ &= 0 \end{aligned} \tag{42}$$

where we have used the Virasoro constraints for the tachyon state to go to the second line, equation (21) to get to the third line, and the identity $\alpha_0^\mu = l_s p^\mu$. To satisfy this we are force to choose the transverse polarisation condition

$$\zeta \cdot k = 0 \tag{43}$$

the Virasoro conditions for $m \geqslant 2$ are trivially satisfied. At first sight, the condition above seems to imply that we have 25 independent polarisations. However, it turns out there are 24 physical states at level one, corresponding to the massless representations of the Poincaré group. This is because we have an underlying gauge invariance, with a physical state with polarisation vector ζ_μ and momentum k^μ being equivalent to a physical state with polarisation $\zeta_\mu + \lambda k^\mu$ and the same momentum, where λ is an arbitrary constant. The extra term is associated with null spurious states; physical states with zero norm which are orthogonal to all physical states. The component of ζ parallel to the momentum do not contribute to any physical quantities and so we do in fact have only 24 independent components, which is what we expect for a photon in 26 dimensions.

(iii) $N = 2$. At level two, the first massive level, we have a spin two particle, which we can write as

$$|G(\epsilon_{\mu\nu}, k\rangle = (\zeta_{\mu\nu} \alpha^\mu_{-1} \alpha^\nu_{-1} + \zeta_\mu \alpha^\mu_{-2}) |0; k\rangle \tag{44}$$

The mass shell condition gives $m^2 = -k^2 = 2l_s^{-2}$. The action of L_m on the (43) gives

$$L_m \big(\zeta_{\mu\nu} \alpha^\mu_{-1} \alpha^\nu_{-1} + \zeta_\mu \alpha^\mu_{-2}\big) |0; k\rangle = \Big(\zeta_\mu \nu \big(\alpha^\mu_{m-1} \alpha^\nu_{-1} + \alpha^\mu_{-1} \alpha^\nu_{m-1}\big) + 2\zeta_\mu \alpha^\mu_{m-2}\Big) |0; k\rangle = 0 \tag{45}$$

where the action of the Virasoro operators on the tachyon state and equation (21) have again been used. For the case of L_1 this gives

$$\left(l_s\zeta_{\mu\nu}\big(\alpha_{-1}^{\nu}k^{\mu}+\alpha_{-1}^{\mu}k^{\nu}\big)+2\zeta_{\mu}\alpha_{-1}^{\mu}\right)|0;k\rangle=0 \tag{46}$$

this gives the following condition

$$l_s\zeta_{(\mu\nu)}k^{\nu}+\zeta_{\mu}=0 \tag{47}$$

The L_2 condition becomes

$$\left(\zeta_{\mu\nu}\eta^{\mu\nu}+2\zeta_{\mu}l_sk^{\mu}\right)|0;k\rangle=0 \tag{48}$$

which gives

$$\zeta_{\mu\nu}\eta^{\mu\nu}+2l_s\zeta_{\mu}k^{\mu}=0 \tag{49}$$

These 27 conditions that we have, in the covariant approach, imply that we have $\frac{D(D+1)}{2}-1=350$ independent components. These are massive, and so they must correspond to irreducible representations of $SO(25)$. However, as we previously outlined in section 6, in the lightcone gauge the only states are

$$\alpha_{-2}^{i}|0;k\rangle\,,\quad \alpha_{-1}^{i}\alpha_{-1}^{j}|0;k\rangle \tag{50}$$

where the index $i\in[1,25]$. This implies that there are $\frac{1}{2}D(D-1)-1=324$ states. Therefore there must be D physical null states in the covariant analysis, corresponding to a massive vector and a massive scalar. These are of the form:

$$\begin{aligned}|\phi_1\rangle&=\chi_{\mu}L_{-1}\alpha_{-1}^{\mu}|0;k\rangle\\|\phi_2\rangle&=(L_{-2}+\frac{3}{2}L_{-1}^{2})|0;k\rangle\end{aligned} \tag{51}$$

These states are orthogonal to all physical states. Demanding they also need to be physical, making them null spurious states, gives $k\cdot\chi=0$ for $|\phi_1\rangle$, whilst $|\phi_2\rangle$ can be shown to automatically satisfy the constraints for $D=26$.

We can then decompose the level two physical states into three categories:
(a) 324 $(=\frac{1}{2}D(D+1)-(D+1))$ positive norm states

$$\zeta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|0;k\rangle \tag{52}$$

where the polarisation tensor satisfies

$$\zeta_{\mu\nu}=\zeta_{\nu\mu},\quad l_sk^{\mu}\zeta_{\mu\nu}=\zeta_{\mu\nu}\eta^{\mu\nu}=0 \tag{53}$$

This set of states corresponds to the Young diagram $(1,1)$ of $SO(25)$.
(b) 25 $(=D-1)$ null spurious states

$$\left(\zeta_\mu \alpha^\mu_{-2} - \frac{2}{l_s k^2} k_\mu \zeta_\nu \alpha^\mu_{-1} \alpha^\nu_{-1}\right) |0; k\rangle \tag{54}$$

with the constraint

$$k^\mu \zeta_\mu = 0 \tag{55}$$

(c) 1 null spurious state

$$\left(l_s k_\mu \alpha^\mu_{-2} + \left[\frac{1 - 2k^2 l_s^2}{D-1}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \frac{k_\mu k_\nu}{k^2}\right] \alpha^\mu_{-1} \alpha^\nu_{-1} \right) |0; k\rangle \tag{56}$$

Thus we find that the physical states form the appropriate Lorentz multiplets as we expected. It is apparent that the positive norm states form complete irreducible representations of $SO(25)$, and so we proceed to list only the positive norm states.

(iv) $N = 3$: At level 3 the mass condition gives $m^2 = -k^2 = 4l_s^{-2}$ and there are two sets of positive norm physical states which are given by the following:

$$\zeta_{\mu\nu\rho} \alpha^\mu_{-1} \alpha^\nu_{-1} \alpha^\rho_{-1} |0; k\rangle \tag{57}$$

where the physical state conditions restrict the polarisation tensor to satisfy the following constraints

$$\zeta_{\mu\nu\rho} = \zeta_{\mu\nu\rho}, \quad \zeta_{\mu\nu\rho} k^\mu l_s = \eta^{\mu\nu} \zeta_{\mu\nu\rho} = 0 \tag{58}$$

Therefore this polarisation tensor has 2900 independent components, corresponding to the representation of $SO(25)$ given by the Young diagram $(1, 1, 1)$.
The second set of physical states are given by

$$\zeta_{\mu\nu} \alpha^\mu_{-1} \alpha^\nu_{-2} |0; k\rangle \tag{59}$$

and the physical state conditions restrict the polarisation tensor to satisfy

$$\zeta_{\mu\nu} = -\zeta_{\nu\mu}, \quad \zeta_{\mu\nu} k^\mu l_s = \eta^{\mu\nu} \zeta_{\mu\nu} = 0 \tag{60}$$

thus the polarisation tensor has 300 independent components and corresponds representation of $SO(25)$ given by the Young diagram (2). This and higher level results can be found in [2] (using $l_s = 1$).

Throughout this section it is clear that the Virasoro operators, through the physical state conditions, play an important role. Using these operators all physical states can be found by solving the set of linear algebraic equations corresponding to each Virasoro constraint.

8 Vertex Operators

In this section we discuss the construction of vertex operators for the bosonic string, using [3], [6] and [4] as primary sources. Vertex operators are operators that represent the emission or absorption of on-shell string modes from a specific point of the string world sheet. They are generalisations of the notion of interaction vertices from field theory, with the generalisation of propagators given by

$$\Delta = (L_0 - 1)^{-1} = \int_0^\infty d\tau e^{-\tau(L_0-1)} \tag{61}$$

where τ represents imaginary time and $L_0 - 1$ represents the 'Hamiltonian'. A general tree amplitude for physical states might then be given by

$$A_M = g^{M-2} \langle\phi_1| V_2(k_2)\Delta V_3(k_3)\cdots\Delta V_{M-1}(k_{M-1})|\phi_M\rangle \tag{62}$$

Thus, to understand objects such as these it is necessary to understand vertex operators. We shall see that the Virasoro operators play a crucial role in the definition of these quantities.

Stringent requirements are placed on the vertex operators via their conformal dimension. Under a general conformal transformation such that $\tau \to \tau'(\tau)$ a local field, $V(\tau)$ of conformal dimension J transforms as

$$V'(\tau') = \left(\frac{d\tau}{d\tau'}\right)^J V(\tau) \tag{63}$$

Under an infinitesimal transformation $\tau \to \tau' = \tau + \epsilon(\tau)$ this becomes

$$\delta V(\tau) = -\epsilon\frac{dV}{d\tau} - JV\frac{d\epsilon}{d\tau} \tag{64}$$

The Virasoro operators generate these conformal transformations with $\epsilon = ie^{im\tau}$. This means that

$$[L_m, V(\tau)] = e^{im\tau}\bigg(-i\frac{d}{d\tau} + mJ\bigg)V(\tau) \tag{65}$$

Vertex operators are used in calculations of scattering amplitudes since, in any conformal field theory, the use of vertex operators ensures that the final results are conformally invariant. For string theory, in particular, we use integrated vertex operators to allow for all possible particle emissions or absorptions on the world-sheet:

$$V = \int d\tau V(\tau) \tag{66}$$

where $V(\tau)$ is a local operator, as we previously described. To compensate for the conformal transformation of the integration measure $d\tau \to d\tau'$ we need to impose the condition that the vertex operator $V(\tau)$ transform with conformal dimension one:

$$V(\tau) \to \left(\frac{d\tau}{d\tau'}\right) V(\tau) \tag{67}$$

In [2] Saskaki and Yamanaka proved that there exists a one to one correspondence between a physical state and a vertex operator. This is a well known result and essential for the construction of amplitudes. We will demonstrate this correspondence for levels zero and one using the old covariant quantisation approach. We follow the approach described in [3] and determine the restrictions that must be imposed on the relevant vertex operator for it to have conformal dimension one.

We start by splitting the mode expansion (8) for $\sigma = 0$ into an "annihilation" part, a "zero" part and a "creation" part:

$$X^\mu = X_-^\mu + X_0^\mu + X_+^\mu \tag{68}$$

$$\begin{aligned} X_+^\mu &= l_s \sum_{n=1}^{\infty} \frac{i}{n} \alpha_n^\mu e^{-in\tau} \\ X_0^\mu &= x^\mu + l_s^2 p^\mu \tau \\ X_-^\mu &= -l_s \sum_{n=1}^{\infty} \frac{i}{n} \alpha_{-n}^\mu e^{in\tau} \end{aligned} \tag{69}$$

The commutation relations for the oscillators and the string coordinates are

$$[\alpha_n^\mu, X^\nu(\tau)] = -i l_s e^{in\tau} \eta^{\mu\nu} \tag{70}$$

The commutation relations for the Virasoro operators and the string coordinates are

$$\begin{aligned} [L_m, X_+^\mu(\tau)] &= -i l_s \sum_{n=m+1}^{\infty} e^{i(m-n)\tau)} \alpha_n^\mu \\ [L_m, X_0^\mu(\tau)] &= -i l_s \alpha_m^\mu \\ [L_m, X_-^\mu(\tau)] &= -i l_s \sum_{n=-\infty}^{m-1} e^{i(m-n)\tau} \alpha_n^\mu \end{aligned} \tag{71}$$

We consider vertex operators $V(k, \sigma = 0, \tau) = V(k, \tau)$, i.e. for emission or absorption at time τ and $\sigma = 0$ on the open string. The simplest candidate for a vertex operator to consider is $\exp(ik \cdot X(0, \tau)$, corresponding to a change in the string state by momentum k^μ. This needs to be normal ordered, so we define the corresponding (lowest order) vertex operator as

$$V_T(k, \tau) =: e^{ik \cdot X(0,\tau)} := e^{ik \cdot X_-} e^{ik \cdot (X_0 + X_+)} \tag{72}$$

The splitting of the exponential operator in the last equality is valid since the each distinct part of the string coordinates commute because they correspond to the sums of different oscillators. To be a valid vertex operator it must have conformal dimension $J = 1$, we compute this using the Virasoro operators.

$$[L_m, V_T] = e^{ik\cdot X_-}[L_m, e^{ik\cdot(X_0+X_+)}] + [L_m, e^{ik\cdot X_-}]e^{ik\cdot(X_0+X_+)} \tag{73}$$

To evaluate this we use the following identities

$$[L_m, e^{in\cdot X_-}] = e^{im\tau}e^{ik\cdot X_-}\left[l_s\sum_{n=1}^{m-1}(k\cdot\alpha_n)e^{-in\tau} + k\cdot(X_0+X_-) + \frac{1}{2}l_s^2k^2(m-1)\right] \tag{74}$$

and

$$[L_m, e^{ik\cdot(X_0+X_+)}] = e^{im\tau}\left[l_s\sum_{n=m}^{\infty}(k\cdot\alpha_n)e^{-in\tau}\right]e^{ik(X_0+X_+)} \tag{75}$$

Plugging this into equation (73) gives

$$\begin{aligned}
&[L_m, V_T] \\
&\quad= e^{im\tau}e^{ik\cdot X_-}\Bigg[l_s\sum_{n=m}^{\infty}(k\cdot\alpha_n)e^{-in\tau} + l_s\sum_{n=1}^{m-1}(k\cdot\alpha_n)e^{-in\tau} \\
&\qquad\qquad + k\cdot(\dot{X}_0+\dot{X}_-) + \frac{1}{2}l_s^2k^2(m-1)\Bigg]e^{ik\cdot(X_0+X_+)} \\
&\quad= e^{im\tau}e^{ik\cdot X_-}\Bigg[l_s\sum_{n=1}^{\infty}(k\cdot\alpha_n)e^{-in\tau} + k\cdot(\dot{X}_0+\dot{X}_-) \\
&\qquad\qquad + \frac{1}{2}l_s^2k^2(m-1)\Bigg]e^{ik\cdot(X_0+X_+)} \\
&\quad= e^{im\tau}e^{ik\cdot X_-}\left[k\cdot\dot{X}_+ + k\cdot(\dot{X}_0+\dot{X}_-) + \frac{1}{2}l_s^2k^2(m-1)\right]e^{ik\cdot(X_0+X_+)} \\
&\quad= e^{im\tau}\Bigg[\big(-i\frac{d}{d\tau}e^{ik\cdot X_-}\big)e^{ik\cdot(X_0+X_+)} + e^{ik\cdot X_-}\big(-i\frac{d}{d\tau}e^{ik\cdot X_0}\big)e^{ik\cdot X_+} \\
&\qquad\qquad + e^{ik\cdot(X_-+X_0)}\big(-i\frac{d}{d\tau}e^{ik\cdot X_+}\big) + \frac{1}{2}ml_s^2k^2V_T\Bigg] \\
&\quad= e^{im\tau}\left[-i\frac{d}{d\tau} + \frac{1}{2}ml_s^2k^2\right]V_T
\end{aligned} \tag{76}$$

Therefore the final result has the right conformal dimension if we choose the conditions $J = 1$ and $l_s^2k^2 = 2$. These are exactly the physical state conditions we previously derived the level one tachyon state for the open bosonic string and so we can tentatively see the link between the two.

For $k^2 = 0$, corresponding to the massless vector meson, (76) shows that choosing the same vertex operator would give an operator with conformal dimension $J = 0$. This isn't appropriate for a vertex operator. However, since the operator $\dot{X}^\mu$ has conformal dimension one, we can rectify this by trying the following vertex operator:

$$V_P(k,\tau) = \frac{1}{l_s}\zeta \cdot \frac{dX}{d\tau} : \exp(ik \cdot X) : \tag{77}$$

Normal ordering this we get

$$V_P = \frac{\zeta_\mu}{l_s}\big(\dot{X}^\mu_- V_T + \dot{X}^\mu_0 V_T + V_T \dot{X}^\mu_+\big) + \frac{\zeta_\mu}{l_s}\big[\dot{X}^\mu_+, V_T\big] \tag{78}$$

We can evaluate the commutator as follows

$$\begin{aligned}
[\dot{X}^\mu_+, \exp(ik \cdot X_-)] &= [l_s \sum_{n=1}^{\infty} \alpha^\mu_n e^{-in\tau}, \exp\big(l_s \sum_{n=1}^{\infty} \frac{1}{m}(k \cdot \alpha_{-m}) e^{im\tau}\big)] \\
&= l_s^2 \sum_{n,m=1}^{\infty} \frac{1}{m} e^{i(m-n)\tau} \eta_{\rho\sigma} k^\rho [\alpha^\mu_n, \alpha^\sigma_{-m}] e^{ik\cdot X_-} \\
&= l_s^2 k^\mu \sum_{n=1}^{\infty} 1 e^{ik\cdot X_-} \\
&= l_s^2 k^\mu \zeta(0) e^{ik\cdot X_-} \\
&= -\frac{1}{2} l_s^2 k^\mu \exp(ik \cdot X_-)
\end{aligned} \tag{79}$$

where we have used the commutation relation given in (16) and evaluated the sum using the regularised zeta function identity $\zeta(0) = -\frac{1}{2}$.
This gives

$$\begin{aligned}
V_P &= \frac{\zeta_\mu}{l_s}\big(\dot{X}^\mu_- V_T + \dot{X}^\mu_0 V_T + V_T \dot{X}^\mu_+\big) - \frac{l_s}{2}(\zeta \cdot k) V_T \\
&= \frac{\zeta_\mu}{l_s} V^\mu_{P'} - \frac{l_s}{2}(\zeta \cdot k) V_T
\end{aligned} \tag{80}$$

The normal ordered vertex operator isn't just proportional to $: \dot{X}^\mu V_T :$ as we might have otherwise expected. To find the vertex operator conditions to give the correct conformal dimension we proceed once more by finding the commutator of the Virasoro operators with each part of the expression for V_P given in (80).

$$
\begin{aligned}
&[L_m, V^\mu_{P'}] \\
&\quad = e^{ik\cdot X_-}\dot{X}^\mu[L_m, e^{ik\cdot(X_0+X_+)}] + [L_m, e^{ik\cdot X_-}]\dot{X}^\mu e^{ik\cdot(X_0+X_+)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + e^{ik\cdot X_-}[L_m, \dot{X}^\mu]e^{ik\cdot(X_0+X_+)} \\
&\quad = e^{im\tau}e^{ik\cdot X_-}\Big[l_s\dot{X}^\mu\sum_{n=m}^{\infty}(k\cdot\alpha_n)e^{-in\tau} + l_s\sum_{n=1}^{m-1}(k\cdot\alpha_n)e^{-in\tau}\dot{X}^\mu \\
&\quad + \big(k\cdot(\dot{X}_0+\dot{X}_-) - \frac{1}{2}l_s^2k^2\big)\dot{X}^\mu + \frac{1}{2}ml_s^2k^2\dot{X}^\mu + m\dot{X}^\mu - i\ddot{X}^\mu\Big]e^{ik\cdot(X_0+X_-)}
\end{aligned}
\tag{81}
$$

By using equation (70) we can switch the order of $(k\cdot\alpha_n)$ and $\dot{X}^\mu$ in the second term:

$$
[\alpha^\mu_n, \dot{X}^\nu] = \frac{d}{d\tau}[\alpha_n\mu, X^\nu(\tau)] = nl_se^{in\tau}\eta^{\mu\nu} \tag{82}
$$

This gives

$$
\begin{aligned}
&[L_m, V^\mu_{P'}] \\
&\quad = e^{im\tau}e^{ik\cdot X_-}\Big[l_s\dot{X}^\mu\sum_{n=1}^{\infty}(k\cdot\alpha_n)e^{-in\tau} + \big(k\cdot(\dot{X}_0+\dot{X}_-) - \frac{1}{2}l_s^2k^2\big)\dot{X}^\mu \\
&\qquad + m\big(\frac{1}{2}l_s^2k^2+1\big)\dot{X}^\mu - i\ddot{X}^\mu + \frac{1}{2}l_s^2m(m-1)k^\mu\Big]e^{ik\cdot(X_0+X_-)} \\
&\quad = e^{im\tau}\Big[e^{ik\cdot X_-}\dot{X}^\mu(k\cdot\dot{X}_+)e^{ik\cdot(X_0+X_+)} + e^{ik\cdot X_-}\big(k\cdot(\dot{X}_0+\dot{X}_-) - \frac{1}{2}l_s^2k^2\big) \\
&\qquad + \dot{X}^\mu e^{ik\cdot(X_0+X_+)} + e^{ik\cdot X_-}(-i\ddot{X}^\mu)e^{ik\cdot(X_0+X_+)} \\
&\qquad + m\big(\frac{1}{2}l_s^2k^2+1\big)V^\mu_{P'} + \frac{1}{2}l_s^2(m^2-m)k^\mu V_T\Big] \\
&\quad = e^{im\tau}\Big[\big[-i\frac{d}{d\tau} + m\big(\frac{1}{2}l_s^2k^2+1\big)\big]V^\mu_{P'} + \frac{1}{2}l_s^2(m^2-m)k^\mu V_T\Big]
\end{aligned}
\tag{83}
$$

Combining this with equation (76) we find that the commutator of the Virasoro generators with the operator V_P is given by

$$
[L_m, V_p] = e^{im\tau}\Big[\big[-i\frac{d}{d\tau} + m\big(\frac{1}{2}l_s^2k^2+1\big)\big]V_P + \frac{1}{2}l_s^2m^2(\zeta\cdot V_T\Big] \tag{84}
$$

Comparing this with equation (64) we find that if we want a vertex operator of conformal dimension $J = 1$ we must choose

$$
\frac{1}{2}l_s^2k^2 = 0, \quad \zeta\cdot k = 0 \tag{85}
$$

These correspond to the generalised mass shell condition and the transverse polarisation condition that arise for the level one massless vector meson state.

This analysis can be continued for higher level states, but the conformal algebra becomes extremely tricky. Further analysis can be found in [3] where an account is given for the conditions that correspond to the massive level two excited state.

In summary, we find the following correspondence between bosonic open string states and the corresponding vertex operators:

$$\begin{array}{rll} \text{Tachyon :} & |0;k\rangle & :e^{ik\cdot X}: \\ \text{Photon :} & \zeta\cdot\alpha_{-1}\,|0;k\rangle & \zeta\cdot\dot{X}:e^{ik\cdot X} \end{array} \tag{86}$$

9 Conclusion

In this report we aimed to identify and discuss the roles of the Virasoro operators in string theory and to an extent we succeeded in this pursuit. We started by discussing the foundations of the Virasoro operators and algebra for the open bosonic string. We then sought to give a clear account of what the Virasoro operators are and what they represent both with and without the central extension.

The relation between the Virasoro operators and the physical state conditions that physical states of the quantum string must satisfy were explained. The operators play a crucial role in the determination of the string spectrum and this was demonstrated through the explicit use of the operators in the construction of several low lying physical states. We further showed that the positive norm physical states correspond to Lorentz multiplets, a result we anticipated due to the fact that the operators commute with the generators of the Poincaré group.

Finally, we analysed the role of the Virasoro operators in the theory of vertex operators. We showed how the Virasoro operators have a very important part in their construction through the definition of the conformal dimension of the relevant vertex operators. Following this we derived explicit expressions for the vertex operators corresponding to emission of a tachyon and a massless vector meson. Once again, throughout this process, the Virasoro operators could be seen to play a very important role in the construction of the state conditions that the vertex operators must satisfy. We concluded this by noting how these vertex operators correspond to physical states, affirming the correspondence between vertex operators and physical for level zero and level one states.

However, whilst we aimed for a "bottom up" approach to the results we obtained in this paper, we were unable to completely avoid making a few assumption. One in particular is the use of space-time dimension $D = 26$ and $a = 1$ which was used repeatedly. The no-ghost theorem proves that this is the appropriate result and the proof of the theorem makes extensive use of the Virasoro operators. In addition, in this paper we focussed our efforts of the theory of the open bosonic string, and in doing so did not analyse the role of the Virasoro operators and algebra for other situations such as the closed string. Furthermore, whilst we discussed how the Virasoro operators are used in various areas of open bosonic string theory, these topics still only represent a small fraction of the myriad of areas in which the Virasoro operators play an important role. Thus, further work could include a detailed discussion of the no-ghost theorem, emphasis of the role of the operators in string theories other than the case of the bosonic open string and a more extensive review of the role of the Virasoro operators in string theory including a wider variety of areas of study.

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