CO51: Rocket Motion

A. Kulanthaivelu
University College

1 Abstract

The aim of this experiment was to determine the trajectories of a rocket being fired at the Moon under different circumstances. The first two scenarios looked at situations where the Moon was fixed on the y-axis of a coordinate system with the centre of the Earth as its origin, and in the third the Moon was modelled as rotating. In the first scenario the rocket was fired from the y-axis, in the second the x-axis and in the third the y-axis again, all from the surface of the Earth. In the latter two cases the angle required to fire the rocket at to hit the moon was determined to be 0.89 and 0.91 radians to the horizontal respectively.

2 Introduction

The motion of a rocket is typically affected by a range factors. We assume the motion of the rocket fired from Earth and the motion of the moon and the rocket is coplanar, the Earth's atmosphere has a negligible effect on the motion and we disregard the possible changing mass of the rocket.
The force on a rocket moving within the gravitational field of the moon and the earth is given by

$$\boldsymbol{F}(\boldsymbol{r}) = -GM_eM_r\frac{1}{|\boldsymbol{r}_r|^2}\frac{\boldsymbol{r}_r}{|\boldsymbol{r}_r|} - GM_mM_r\frac{1}{|\boldsymbol{r}_r-\boldsymbol{r}_m|^2}\frac{(\boldsymbol{r}_r-\boldsymbol{r}_m)}{|\boldsymbol{r}_r-\boldsymbol{r}_m|} \tag{1}$$

Therefore its acceleration is

$$\boldsymbol{a}(\boldsymbol{r}) = \frac{\boldsymbol{F}(\boldsymbol{r})}{M_r} = -GM_e\frac{1}{|\boldsymbol{r}_r|^2}\frac{\boldsymbol{r}_r}{|\boldsymbol{r}_r|} - GM_m\frac{1}{|\boldsymbol{r}_r-\boldsymbol{r}_m|^2}\frac{(\boldsymbol{r}_r-\boldsymbol{r}_m)}{|\boldsymbol{r}_r-\boldsymbol{r}_m|} \tag{2}$$

Resolving this into x and y components we get

$$\begin{aligned} a_x &= -GM_e\frac{1}{|\boldsymbol{r}_r|^2}\frac{x_r}{|\boldsymbol{r}_r|} - GM_m\frac{1}{|\boldsymbol{r}_r-\boldsymbol{r}_m|^2}\frac{(x_r-x_m)}{|\boldsymbol{r}_r-\boldsymbol{r}_m|} \\ a_y &= -GM_e\frac{1}{|\boldsymbol{r}_r|^2}\frac{y_r}{|\boldsymbol{r}_r|} - GM_m\frac{1}{|\boldsymbol{r}_r-\boldsymbol{r}_m|^2}\frac{(y_r-y_m)}{|\boldsymbol{r}_r-\boldsymbol{r}_m|} \end{aligned} \tag{3}$$

We integrate these equations with respect to time using the Euler method in the first two cases and the improved Euler method in the last case to plot the trajectory of the rocket and calculate the angle the rocket needs to be fired at to hit the moon (in the latter two cases only).
In the first case we plotted the path of the rocket fired at the moon, which was fixed on our y-axis at a distance of 222 moon-radii from Earth, at an angle of 89.9 degrees to the horizontal which we took to be our x-axis.
In the second scenario we fired the rocket from the x axis; in this case we determined the angle at which the rocket needed to be fired at in order to reach the moon, which was fixed on the y-axis as before.
In the third scenario the moon was modelled as following the orbital equation;

$$\boldsymbol{r}_m = R_0[\cos(\Omega t),\ \sin(\Omega t)] \quad (4)$$

Here we fired the rocket from the y-axis and once again calculated the angle at which the rocket needed to be fired at in order to hit the moon and plotted its path.
Constant distances and masses such as the mass of the earth and radius of the moon are expressed in moon-masses and moon-radii as depicted in Figure 1 so that the calculations don't go beyond the range of the computer. In all three cases the rocket was fired with a speed of 0.0066 moon radii per second.

M_M (Mass of the Moon)	1.0 moon-mass
M_E (Mass of the Earth)	83.3 moon-mass
R_M (Radius of the Moon)	1.0 moon-radius
R_E (Radius of the Earth)	3.7 moon-radii
G (Gravitation constant)	9.63×10^{-7} moon-radii^{3}moon-masses^{-1}s^{-1}

Figure 1: Table showing relevant constants expressed in Moon dimensions

3 Solution

Part 1 – The simple stationary case

In this case the equations were solved using the Euler method which is stated as:

$$x_{n+1} \approx x_n + \Delta t f(x_n, t_n) \quad (5)$$

where $f(x,t) = \frac{dx}{dt}$ and Δt is the step size (which in this case is chosen to be 10s) such that the total time $t = n\Delta t$.
In the first case, applying this method to our acceleration equations we get:

$$\begin{aligned} v_{x(t+\Delta t)} &= v_{x(t)} + \Delta t a(x) \\ v_{y(t+\Delta t)} &= v_{y(t)} + \Delta t a(y) \end{aligned} \quad (6)$$

where the initial velocities, $v_{x(0)}$ and $v_{y(0)}$, are $v_0 \cos\theta$ and $v_0 \sin\theta$ respectively. In the first case θ was fixed at 89.9° to the horizontal but in the subsequent cases the value of θ needed to land on the moon was to be determined. Once the velocity had been calculated the Euler method was used once again to find the position as follows:

$$\begin{aligned} x_{t+\Delta t} &= x_t + \Delta t v_x \\ y_{t+\Delta t} &= y_t + \Delta t v_y \end{aligned} \quad (7)$$

where $(x_0, y_0) = (0,3.7)$ in the first case and $(3.7, 0)$ in the second case.

Part 2 – The orbiting moon

In this case the equations were solved using the improved Euler method which is stated as:

$$x_{n+1} \approx x_n + \frac{1}{2}\Delta t[f(x_n, t_n) + f(x_{n+1}, t_{n+1})] \quad (8)$$

which can be rewritten using the first approximation equation (5) as

$$x_{n+1} \approx x_n + \frac{1}{2}\Delta t[f(x_n, t_n) + f(x_n + \Delta t f(x_n, t_n),\ t_{n+1})] \tag{9}$$

We first calculated a new set of values (x', y') with equations (7) using the Euler method and then calculated a new set of values (v'_x, v'_y) using the (x, y) values and calculated (x', y') values such that:

$$\begin{aligned} v'_{x(t+\Delta t)} &= v_{x(t)} + \frac{1}{2}\Delta t(a(x,y) + a(x',y')) \\ v'_{y(t+\Delta t)} &= v_{y(t)} + \frac{1}{2}\Delta t(a(x,y) + a(x',y')) \end{aligned} \tag{10}$$

Finally we used the improved Euler method to calculate a new value of (x, y) as follows

$$\begin{aligned} x_{t+\Delta t} &= x_t + \frac{1}{2}\Delta t(v'_x + v_x) \\ y_{t+\Delta t} &= y_t + \frac{1}{2}\Delta t(v'_y + v_y) \end{aligned} \tag{11}$$

where $(x_0, y_0) = (0{,}3.7)$.

To determine the angle to fire the rocket at in order for it to hit the moon the procedure was repeated over angles ranging from 0 to $\frac{\pi}{2}$ in increments of 0.01 radians over a time period of 200000 seconds within which time the rocket would typically hit the moon. If the rocket hit the moon within this time period the angle at which it was fired was recorded. Finally in all cases the calculations were stopped once the distance of the rocket to the centre of the moon became less than the radius of the moon, indicating it had hit the moon.

4 Results

In the first case we solved equations (3) to obtain the graph (Figure 2) showing the trajectory of the rocket fired at the moon, fixed on the y-axis at a position (0, 222), from the position $(0{,}3.7)$. In the second case the rocket was fired at the moon, still fixed on the y-axis, from the position $(3.7, 0)$ and the angle required to hit the moon was found to be 0.89 radians to two decimal places. We also obtained another graph (Figure 3) showing its trajectory. Finally, in the third case, we fired the rocket from position $(3.7, 0)$ once more and the angle required to hit the orbiting moon was determined and it was found to be 0.91 radians to two decimal places and the graph (Figure 4) was obtained showing the trajectory of the rocket. In all graphs the rocket is fired from a point on the surface of the Earth and finishes at a point on the surface of the moon.

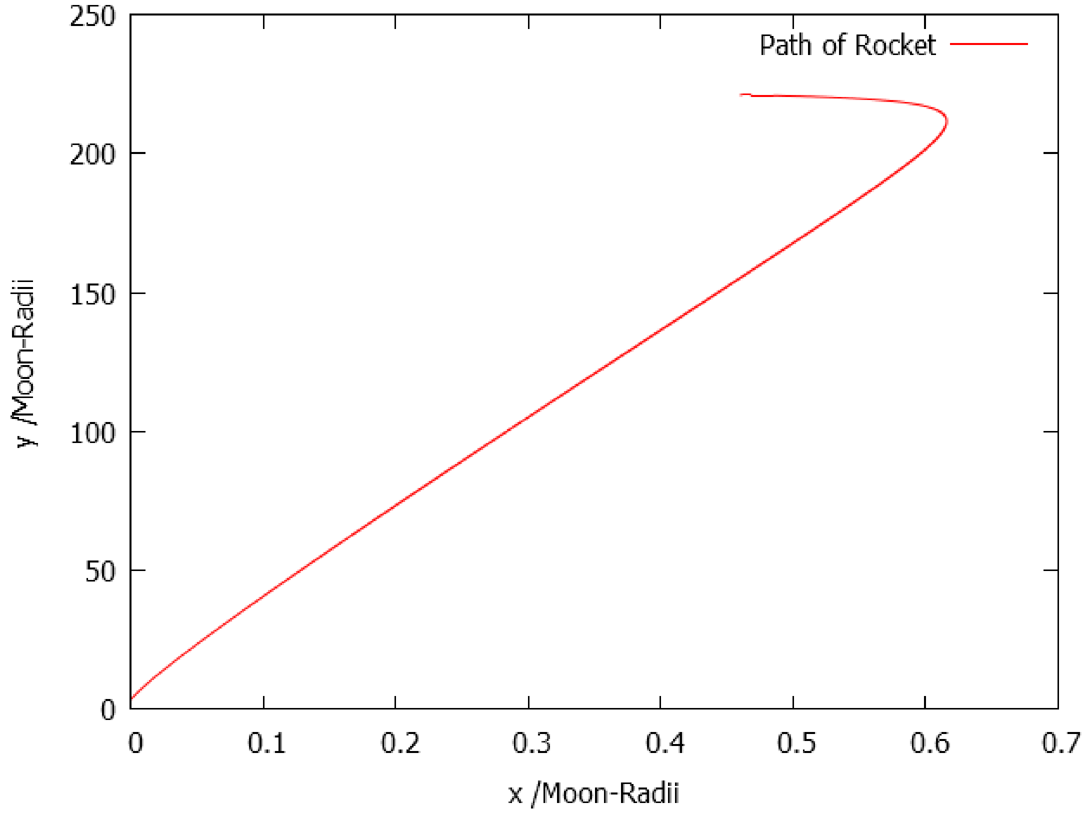

Figure 2: Trajectory of rocket fired at the moon from a point (0, 3.7) at an angle of 89.9° to the horizontal

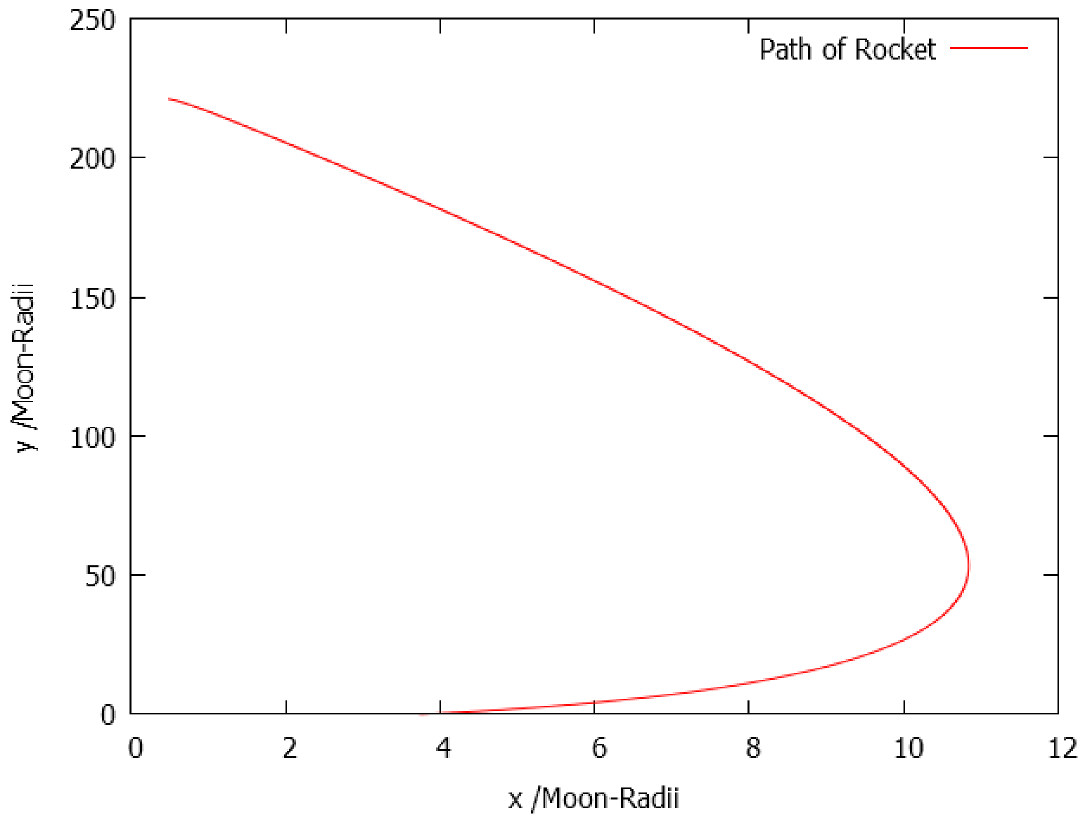

Figure 3: Trajectory of rocket fired at the moon from a point (3.7, 0) at an angle of 0.89 radians to the horizontal

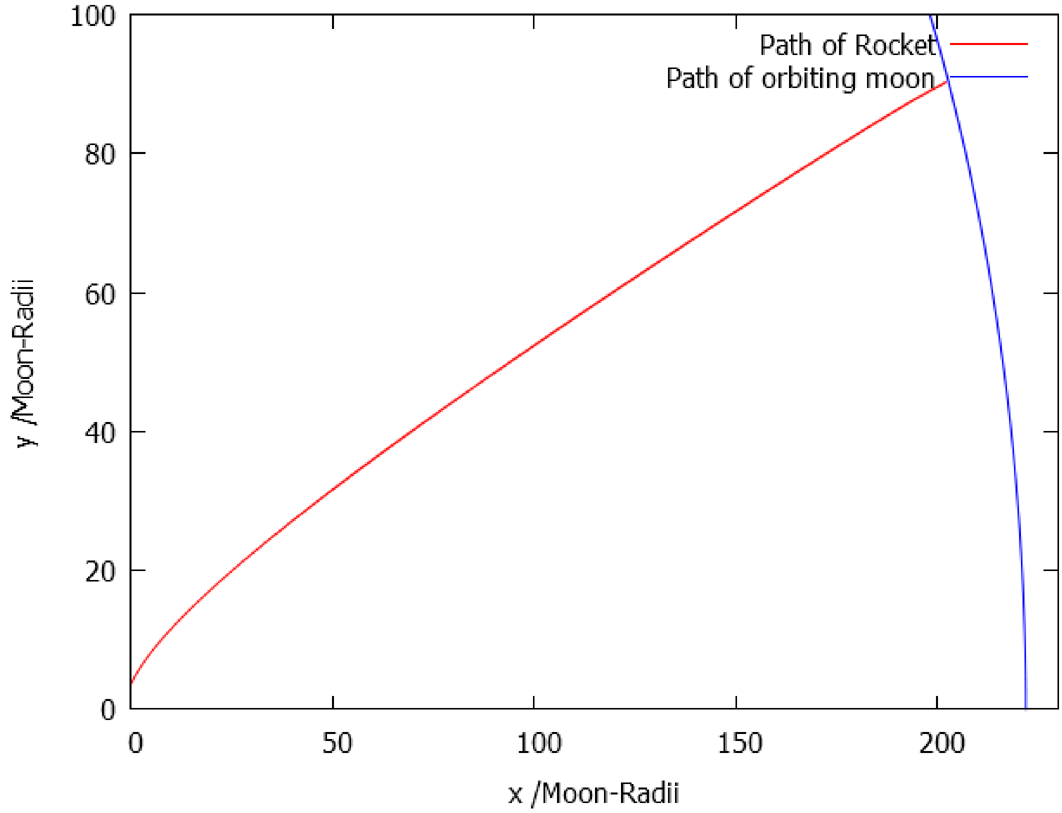

Figure 4: Trajectory of rocket fired at an orbiting moon from a point (0, 3.7) an angle of 0.91 radians to the horizontal

5 Conclusion

Using the three computer models the required launch angle to fire the rocket at in order to hit the moon was found to be 0.89 radians and 0.91 radians when fired from the x-axis in the stationary case and the y-axis in the orbiting moon case respectively and the trajectories of the rocket fired under all three circumstances has been determined, following the space-curves depicted on figures 2-4. In the first plot the rocket shows the expected behaviour, travelling almost vertically upwards, with a small horizontal deviation to the right due to its small horizontal velocity component and then curves at it nears the moon and experiences a greater attractive force from it. In the second model the path of the rocket is noticeably curved due to it being fired from the x axis but still collides with the moon when fired with a launch angle of 0.89 radians to the horizontal. Finally in the third plot the rocket is fired with a launch angle of 0.91 radians and after a small but rapid deviation to the left travels almost directly to the position at which it collides with the rotating moon.

Nevertheless our model contains several simplifying assumptions. For instance:

- The rocket is assumed to have a constant mass throughout its motion and to start with an initial non-zero velocity. In reality a rocket would have changing mass as it burns fuel to accelerate itself upwards or loses booster rockets and it would also start off at zero velocity. This could be accounted for by modifying the derived equation for the acceleration of the rocket to account for changing mass, modifying the initial equations of motion as the rocket accelerates away from the surface of the Earth until the rocket remains at constant mass.

- The only forces acting on the rocket were assumed to be the gravitational forces of the Earth and the Moon whereas there are bound to be other forces acting on it as well. For example the rocket may be subject to air resistance whilst travelling through the Earth's atmosphere. In fact the rocket is assumed to have negligible dimensions; however the dimensions of the rocket could affect the rocket's motion through the Earth's atmosphere in terms of aerodynamic drag forces. Furthermore the rocket would be subject to gravitational forces from other celestial bodies such as the sun for example. This may be accounted for by adding on another term in the equation of motion for the rocket to model the drag forces on the rocket and by adding further terms for the gravitational attraction of the sun on the rocket.

- The moon is fixed in one position in the first two scenarios and assumed to be rotating in a circular orbit for the third. In reality the moon would be travelling in an ellipse with the Earth at one focus. Modifying the equation for the orbital of the moon so that it moves in an ellipse and thus in a more realistic and accurate way would remedy this.

- The motion of the rocket and the moon is modelled as being coplanar, moving in only in two dimensions. However in reality they may not move in the same plane but two completely separate planes. Adding an extra dimension to make the motion three dimensional would remedy this and allow our model to reflect reality more closely.

Modifying these assumptions would make the model more realistic and could extend its use to predicting the trajectories of rockets in different circumstances and calculate other parameters in addition to launch angle.

Appendix A Source Code for Part 1 i)

The source code for the first part of the first problem (where the Moon is stationary and the rocket is fired from the y-axis) is:

```
/****************************************************
*  C051 ROCKET SCIENCE - Part 1 i)
*  A program designed to plot the trajectory of
*  a rocket on a mission to land on the Moon
****************************************************/

#include <math.h>
#include <stdio.h>
#include <stdlib.h>

const double G=9.63E-7;                /*Constants used throughout the program*/
const double Mm=1.0;
const double Me=83.3;
const double Rm=1.0;
const double Re=3.7;
const double v0=0.0066;
const double theta=(89.9*(M_PI))/180 ;
const double xm=0.0;
const double ym=222.0;

/*Function determining the x component of the acceleration for given set of x and y*/
double ax(double x, double y)
{ double Rr, Rrm;
  Rr=sqrt((x*x)+(y*y));
  Rrm=sqrt((x-xm)*(x-xm) + (y-ym)*(y-ym));
  return -((G*Me*x)/(Rr*Rr*Rr)) - ((G*Mm*(x-xm))/(Rrm*Rrm*Rrm));
}

/*Function determining the y component of the acceleration for given set of x and y*/
double ay(double x, double y)
{ double Rr, Rrm;
  Rr=sqrt((x*x)+(y*y));
  Rrm=sqrt((x-xm)*(x-xm) + (y-ym)*(y-ym));
  return -((G*Me*y)/(Rr*Rr*Rr)) - ((G*Mm*(y-ym))/(Rrm*Rrm*Rrm));
}

int main()
{   double Vx, Vy, x, y, t;
    double x_old, y_old, Vx_old, Vy_old;
    double step=10;

    FILE *fout;
    if ((fout=fopen("Rocket1.txt","w"))==NULL)
    {  fprintf(stdout,"Cannot open Rocket1.txt\n");
       exit(EXIT_FAILURE);}

    x_old=0;
    y_old=3.7;
    Vx_old=v0*cos(theta);
    Vy_old=v0*sin(theta);
    t=0.0;

    /*Use the Euler method to solve the equations at each point until it reaches the
Moon*/
    while ( sqrt((x-xm)*(x-xm) + (y-ym)*(y-ym)) > Rm)
    {  Vx=Vx_old + step*ax(x_old,y_old);
```

```
        Vy=Vy_old + step*ay(x_old,y_old);
        x=x_old + step*Vx_old;
        y=y_old + step*Vy_old;
        t=t+step;
        x_old=x;
        y_old=y;
        Vx_old=Vx;
        Vy_old=Vy;

        fprintf(fout,"%f\t %f\n", x, y);}

       fclose(fout);
       exit(0);
       }
```

Appendix B   Source Code for Part 1 ii)

The source code for the second part of the first problem (where the Moon is stationary and the rocket is fired from the x-axis) is:

```
/*****************************************************
*  CO51 ROCKET SCIENCE - Part 1 ii)
*  A program designed to plot the trajectory of
*  a rocket on a mission to land on the Moon
*****************************************************/

#include <math.h>
#include <stdio.h>
#include <stdlib.h>

const double G=9.63E-7;              /*Constants used throughout the program*/
const double Mm=1.0;
const double Me=83.3;
const double Rm=1.0;
const double Re=3.7;
const double v0=0.0066;
const double xm=0.0;
const double ym=222.0;

/*Function determining the x component of the acceleration for given set of x and y*/
double ax(double x, double y)
{ double Rr, Rrm;
  Rr=sqrt((x*x)+(y*y));
  Rrm=sqrt((x-xm)*(x-xm) + (y-ym)*(y-ym));
  return -((G*Me*x)/(Rr*Rr*Rr)) - ((G*Mm*(x-xm))/(Rrm*Rrm*Rrm));
}

/*Function determining the y component of the acceleration for given set of x and y*/
double ay(double x, double y)
{ double Rr, Rrm;
  Rr=sqrt((x*x)+(y*y));
  Rrm=sqrt((x-xm)*(x-xm) + (y-ym)*(y-ym));
  return -((G*Me*y)/(Rr*Rr*Rr)) - ((G*Mm*(y-ym))/(Rrm*Rrm*Rrm));
}

int main()
{   double Vx, Vy, x, y, t, theta, phi;
    double x_old, y_old, Vx_old, Vy_old;
    double step=10.0;
    double Rr, Rrm;
```

```
    FILE *fout;
    if ((fout=fopen("Rocket2.txt","w"))==NULL)
    { printf("Cannot open %s\n", "Rocket2.txt");
    exit(EXIT_FAILURE);}

    /*Scan over a range of theta values from 0 to pi/2
      if the rocket lands on the Moon for any value of
      theta the loop breaks and the value is stored as phi*/
    {for (theta=0; theta <= (M_PI)/2; theta=theta+0.01)
    { x_old=3.7;
      y_old=0.0;
      Vx_old=v0*cos(theta);
      Vy_old=v0*sin(theta);
      t=0.0;

      /*Use the Euler method to solve the equations at each point until it reaches the
Moon or t=200000*/
      for (t=0.0; t<=200000; t+=step)
      {Vx=Vx_old+step*ax(x_old, y_old);
      Vy=Vy_old+step*ay(x_old, y_old);
      x=x_old+step*Vx_old;
      y=y_old+step*Vy_old;
      x_old=x;
      y_old=y;
      Vx_old=Vx;
      Vy_old=Vy;
      if(sqrt((x-xm)*(x-xm)+(y-ym)*(y-ym)) <= 1)
      { phi=theta;
        break;}
        ;}}}

    { x_old=3.7;
      y_old=0.0;
      Vx_old=v0*cos(phi);
      Vy_old=v0*sin(phi);
      t=0.0;

      /*Use the Euler method to solve the equations at each point until it reaches the
Moon for determined value of phi*/
      while(sqrt((x-xm)*(x-xm)+(y-ym)*(y-ym)) > Rm)
      { Vx=Vx_old+step*ax(x_old, y_old);
        Vy=Vy_old+step*ay(x_old, y_old);
        x=x_old+step*Vx_old;
        y=y_old+step*Vy_old;
        x_old=x;
        y_old=y;
        Vx_old=Vx;
        Vy_old=Vy;
        t=t+step;

        fprintf(fout,"%f\t %f\n", x, y);
        }}
     fclose(fout);
     exit(0);
}
```

Appendix C   Source Code for Part 2

The source code for the second problem (where Moon is rotating and the rocket is fired from the y-axis) is:

```
/****************************************************
*  CO51 ROCKET SCIENCE - Part 2
*  A program designed to plot the trajectory of
*  a rocket on a mission to land on the Moon
****************************************************/

#include <math.h>
#include <stdio.h>
#include <stdlib.h>

const double G=9.63E-7;              /*Constants used throughout the program*/
const double Mm=1.0;
const double Me=83.3;
const double Rm=1.0;
const double Re=3.7;
const double v0=0.0066;
const double R0=222.0;
const double w=2.6615E-6;

/*Function determining the x component of the acceleration for given set of x, y and
t*/
double ax(double x, double y, double t)
{ double Rr, Rrm;
  Rr=sqrt((x*x)+(y*y));
  Rrm=sqrt((x-(R0*cos(w*t)))*(x-(R0*cos(w*t))) + (y-(R0*sin(w*t)))*(y-(R0*sin(w*t))));
  return -((G*Me*x)/(Rr*Rr*Rr)) - ((G*Mm*(x-(R0*cos(w*t))))/(Rrm*Rrm*Rrm));
}

/*Function determining the y component of the acceleration for given set of x, y and
t*/
double ay(double x, double y, double t)
{ double Rr, Rrm;
  Rr=sqrt((x*x)+(y*y));
  Rrm=sqrt((x-(R0*cos(w*t)))*(x-(R0*cos(w*t))) + (y-(R0*sin(w*t)))*(y-(R0*sin(w*t))));
  return -((G*Me*y)/(Rr*Rr*Rr)) - ((G*Mm*(y-(R0*sin(w*t))))/(Rrm*Rrm*Rrm));
}

int main()
{   double Vx, Vy, x_new, y_new, t,xm, ym, theta, phi;
    double x_old, y_old, Vx_old, Vy_old, x_prime, y_prime;
    double step=10.0;

    FILE *fout;
    if ((fout=fopen("Rocket3.txt","w"))==NULL)
    { printf("Cannot open %s\n", "Rocket3.txt");
    exit(EXIT_FAILURE);}

    /*Scan over a range of theta values from 0 to pi/2
      if the rocket lands on the Moon for any value of
      theta the loop breaks and the value is stored as phi*/
    {for (theta=0; theta <=(M_PI)/2; theta+=0.01)
    { x_old=0.0;
      y_old=3.7;
      Vx_old=v0*cos(theta);
      Vy_old=v0*sin(theta);
      t=0.0;

     /*Use the Euler method to solve the equations at each point until it reaches the
Moon or t=200000*/
```

```
 for (t=0.0; t<=200000; t+=step)
{ xm=R0*cos(w*t);
  ym=R0*sin(w*t);
  x_prime=x_old+step*Vx_old;
  y_prime=y_old+step*Vy_old;
  Vx=Vx_old+step*(ax(x_old, y_old, t)+ax(x_prime, y_prime, (t+step)))/2;
  Vy=Vy_old+step*(ay(x_old, y_old, t)+ay(x_prime, y_prime, (t+step)))/2;
  x_new=x_old+(step*(Vx_old+Vx)/2);
  y_new=y_old+(step*(Vy_old+Vy)/2);

  x_old=x_new;
  y_old=y_new;
  Vx_old=Vx;
  Vy_old=Vy;

  if(sqrt((x_new-xm)*(x_new-xm) + (y_new-ym)*(y_new-ym)) <= 1)
  { phi=theta;
    break;};
    }}}

 {x_old=0.0;
  y_old=3.7;
  Vx_old=v0*cos(phi);
  Vy_old=v0*sin(phi);
  t=0.0;

  /*Use the Euler method to solve the equations at each point until it reaches the
moon for determined value of phi*/
  while(sqrt((x_new-xm)*(x_new-xm) + (y_new-ym)*(y_new-ym)) > 1)
   { xm=R0*cos(w*t);
     ym=R0*sin(w*t);
     x_prime=x_old+step*Vx_old;
     y_prime=y_old+step*Vy_old;
     Vx=Vx_old+step*(ax(x_old, y_old, t)+ax(x_prime, y_prime, (t+step)))/2;
     Vy=Vy_old+step*(ay(x_old, y_old, t)+ay(x_prime, y_prime, (t+step)))/2;
     x_new=x_old+step*(Vx_old+Vx)/2;
     y_new=y_old+step*(Vy_old+Vy)/2;
     t=t+step;
     x_old=x_new;
     y_old=y_new;
     Vx_old=Vx;
     Vy_old=Vy;

  fprintf(fout,"%f\t %f\n", x_new, y_new);
  }}

  fclose(fout);
  exit(0);
  }
```